

# Special types of intuitionistic fuzzy left $h$ -ideals of hemirings

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**Abstract:** Characteristic, normal and completely normal intuitionistic fuzzy left  $h$ -ideals of hemirings are described.

**Key-Words:** Hemiring, intuitionistic fuzzy left  $h$ -ideal, normal fuzzy set, fuzzy characteristic.

## 1 Introduction

Hemirings, as semirings with zero and commutative addition, appear in a natural manner in some applications to the theory of automata and formal languages (see [1, 9, 12]). It is a well known result that regular languages form so-called star semirings. According to the well known theorem of Kleene, the languages, or sets of words, recognized by finite-state automata are precisely those that are obtained from letters of input alphabets by the application of the operations: sum (union), concatenation (product), and Kleene star (Kleene closure). If a language is represented as a formal series with the coefficients in a Boolean hemiring, then the Kleene theorem can be well described by the Kleen-Schützenberger theorem. Moreover, if the coefficient hemiring is a field, then a series is rational if and only if its syntactic algebra (see [9, 12, 15] for details) has a finite rank. Many other applications with references can be found in [9] and in a guide to the literature on semirings and their applications [6].

Ideals of hemirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals. Many results in rings apparently have no analogues in hemirings using only ideals. Henriksen defined in [7] a more restricted class of ideals in semirings, which is called the class of  $k$ -ideals, with the property that if the semiring  $R$  is a ring then a complex in  $R$  is a  $k$ -ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals, called now  $h$ -ideals, has been given and investigated by Izuka [8] and La Torre [13]. It is interesting that the regularity of hemirings can be characterized by fuzzy  $h$ -ideals [17]. General properties of fuzzy  $k$ -ideals are described in [4, 5, 11]. Other important results connected with fuzzy  $h$ -ideals in hemirings were

obtained in [10] and [16].

## 2 Preliminaries

By a *semiring* is mean an algebraic system  $(R, +, \cdot)$  consisting of a nonempty set  $R$  together with two binary operations on  $R$  called addition and multiplication (denoted in the usual manner) such that  $(R, +)$  and  $(R, \cdot)$  are semigroups satisfying for all  $x, y, z \in R$  the following distributive laws

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz.$$

By a *zero* we mean an element  $0 \in R$  such that  $0x = x0 = 0$  and  $0 + x = x + 0 = x$  for all  $x \in R$ . A semiring with zero and a commutative semigroup  $(R, +)$  is called a *hemiring*.

A nonempty subset  $A$  of  $R$  is said to be a *left ideal* if it is closed with respect to the addition and such that  $RA \subseteq A$ . A left ideal  $A$  is called a *left  $h$ -ideal* (cf. [7]) if for any  $x, z \in R$  and  $a, b \in A$  from  $x + a + z = b + z$  it follows  $x \in A$ .

By a *fuzzy set* of a hemiring  $R$  we mean any mapping  $\mu$  from  $R$  to  $[0, 1]$ . For any mapping  $f$  from  $R$  to  $S$  we can define in  $R$  a new fuzzy set  $\mu^f$  putting  $\mu^f(x) = \mu(f(x))$  for all  $x \in R$ . Clearly  $\mu^f(x_1) = \mu^f(x_2)$  for  $x_1, x_2 \in f^{-1}(x)$ .

For each fuzzy set  $\mu$  in  $R$  and any  $\alpha \in [0, 1]$  we define two sets

$$U(\mu, \alpha) = \{x \in R \mid \mu(x) \geq \alpha\},$$

$$L(\mu, \alpha) = \{x \in R \mid \mu(x) \leq \alpha\},$$

which are called an *upper* and *lower level cut* of  $\mu$  and can be used to the characterization of  $\mu$ . The *complement* of  $\mu$ , denoted by  $\bar{\mu}$ , is the fuzzy set on  $R$  defined by  $\bar{\mu}(x) = 1 - \mu(x)$ .

A fuzzy set  $\mu$  of a hemiring  $R$  is called a *fuzzy left  $h$ -ideal* (cf. [10]) if for all  $a, b, x, z \in R$  the following

three conditions hold:

$$\begin{aligned}\mu(x + y) &\geq \min\{\mu(x), \mu(y)\}, \\ \mu(xy) &\geq \mu(y), \\ x + a + z = b + z &\longrightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}.\end{aligned}$$

As an important generalization of the notion of fuzzy sets, Atanassov introduced in [2] the concept of an *intuitionistic fuzzy set* (IFS for short) defined as objects having the form:

$$A = (\mu_A, \lambda_A) = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in R\},$$

where the fuzzy sets  $\mu_A$  and  $\lambda_A$  denote the *degree of membership* (namely  $\mu_A(x)$ ) and the *degree of non-membership* (namely  $\lambda_A(x)$ ) of each element  $x \in R$  to the set  $A$  respectively, and  $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$  for all  $x \in R$ .

According to [2], for every two intuitionistic fuzzy sets  $A = (\mu_A, \lambda_A)$  and  $B = (\mu_B, \lambda_B)$  in  $R$ , we define:  $A \subseteq B$  if and only if  $\mu_A(x) \leq \mu_B(x)$  and  $\lambda_A(x) \geq \lambda_B(x)$  for all  $x \in R$ . Obviously  $A = B$  means that  $A \subseteq B$  and  $B \subseteq A$ .

### 3 Intuitionistic fuzzy left h-ideals

**Definition 1** An IFS  $A = (\mu_A, \lambda_A)$  on a hemiring  $R$  is called an *intuitionistic fuzzy left h-ideal* (IF left h-ideal for short) if

- (1)  $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\},$
  - (2)  $\lambda_A(x + y) \leq \max\{\lambda_A(x), \lambda_A(y)\},$
  - (3)  $\mu_A(xy) \geq \mu_A(y),$
  - (4)  $\lambda_A(xy) \leq \lambda_A(y),$
  - (5)  $x + a + z = b + z \longrightarrow \mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\},$
  - (6)  $x + a + z = b + z \longrightarrow \lambda_A(x) \leq \max\{\lambda_A(a), \lambda_A(b)\}$
- hold for all  $a, b, x, y, z \in R$ .

An IFS  $A = (\mu_A, \lambda_A)$  satisfying the first four conditions is called an *intuitionistic fuzzy left ideal*.

The family of all intuitionistic fuzzy left h-ideals of a hemiring  $R$  will be denoted by  $IFI(R)$ .

It is not difficult to see that  $\mu_A(x) \leq \mu_A(0)$  and  $\lambda_A(0) \leq \lambda_A(x)$  for each  $A \in IFI(R)$  and  $x \in R$ .

**Example 2** On a four element hemiring  $(R, +, \cdot)$  defined by the following two tables:

+	0	1	2	3	·	0	1	2	3
0	0	1	2	3	0	0	0	0	0
1	1	1	2	3	1	0	1	1	1
2	2	2	2	3	2	0	1	1	1
3	3	3	3	2	3	0	1	1	1

consider an IFS  $A = (\mu_A, \lambda_A)$ , where  $\mu_A(0) = 0.4$ ,  $\lambda_A(0) = 0.2$  and  $\mu_A(x) = 0.2$ ,  $\lambda_A(x) = 0.7$  for all  $x \neq 0$ . It is not difficult to verify that  $A \in IFI(R)$ .

**Example 3** Let  $N$  be the set of all non-negative integers and let

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \langle 4 \rangle, \\ \frac{1}{2} & \text{if } x \in \langle 2 \rangle - \langle 4 \rangle, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\langle n \rangle$  denotes the set of all non-negative integers divided by  $n$ . Then  $(N, +, \cdot)$  is a hemiring and  $A = (\mu, \bar{\mu})$  is its IF left h-ideal.

The following results can be proved by the verification of the corresponding axioms.

**Proposition 4** A fuzzy set  $\mu_A$  is a fuzzy left h-ideal of  $R$  if and only if  $A = (\mu_A, \bar{\mu}_A)$  is an IF fuzzy left h-ideal of  $R$ .

**Proposition 5** An IFS  $A = (\mu_A, \lambda_A)$  is an IF left h-ideal of  $R$  if and only if  $\mu_A$  and  $\lambda_A$  are fuzzy left h-ideals of  $R$ .

**Proposition 6** [3] Let  $A$  be a nonempty subset of a hemiring  $R$ . Then an IFS  $(\mu_A, \lambda_A)$  defined by

$$\begin{aligned}\mu_A(x) &= \begin{cases} \alpha_2 & \text{if } x \in A, \\ \alpha_1 & \text{for } x \notin A, \end{cases} \\ \lambda_A(x) &= \begin{cases} \beta_2 & \text{if } x \in A, \\ \beta_1 & \text{for } x \notin A, \end{cases}\end{aligned}$$

where  $0 \leq \alpha_1 < \alpha_2 \leq 1$ ,  $0 \leq \beta_2 < \beta_1 \leq 1$  and  $\alpha_i + \beta_i \leq 1$  for  $i = 1, 2$ , is an IF left h-ideal of  $R$  if and only if  $A$  is a left h-ideal of  $R$ .

**Definition 7** Let  $A = (\mu_A, \lambda_A)$  be an intuitionistic fuzzy set in a hemiring  $R$  and let  $\alpha, \beta \in [0, 1]$  be such that  $\alpha + \beta \leq 1$ . Then the set

$$R_A^{(\alpha, \beta)} = \{x \in R \mid \alpha \leq \mu_A(x), \lambda_A(x) \leq \beta\}$$

is called an  $(\alpha, \beta)$ -level subset of  $A = (\mu_A, \lambda_A)$ .

The set of all  $(\alpha, \beta) \in \text{Im}(\mu_A) \times \text{Im}(\lambda_A)$  such that  $\alpha + \beta \leq 1$  is called the *image* of  $A = (\mu_A, \lambda_A)$ .

Clearly  $R_A^{(\alpha, \beta)} = U(\mu_A, \alpha) \cap L(\lambda_A, \beta)$ , where  $U(\mu_A, \alpha)$  and  $L(\lambda_A, \beta)$  are upper and lower level subsets of  $\mu_A$  and  $\lambda_A$ , respectively.

**Theorem 8** [3] An IFS  $A = (\mu_A, \lambda_A)$  is an IF left  $h$ -ideal of  $R$  if and only if  $R_A^{(\alpha, \beta)}$  is a left  $h$ -ideal of  $R$  for every  $(\alpha, \beta) \in \text{Im}(\mu_A) \times \text{Im}(\lambda_A)$  such that  $\alpha + \beta \leq 1$ , i.e., if and only if all nonempty level subsets  $U(\mu_A, \alpha)$  and  $L(\lambda_A, \beta)$  are left  $h$ -ideals of  $R$ .

**Theorem 9** [3] Let  $A = (\mu_A, \lambda_A)$  be an intuitionistic fuzzy left  $h$ -ideal of a hemiring  $R$  and let  $x \in R$ . Then  $\mu_A(x) = \alpha$ ,  $\lambda_A(x) = \beta$  if and only if  $x \in U(\mu_A, \alpha)$ ,  $x \notin U(\mu_A, \gamma)$  and  $x \in L(\lambda_A, \beta)$ ,  $x \notin L(\lambda_A, \delta)$  for all  $\gamma > \alpha$  and  $\delta < \beta$ .

## 4 Characteristic IF left $h$ -ideals

**Definition 10** A left  $h$ -ideal  $A$  of a hemiring  $R$  is said to be *characteristic* if  $f(A) = A$  for all  $f \in \text{Aut}(R)$ , where  $\text{Aut}(R)$  is the set of all automorphisms of  $R$ .

**Definition 11** An IFS  $A = (\mu_A, \lambda_A)$  of  $R$  is called an *intuitionistic fuzzy characteristic* if  $\mu_A^f(x) = \mu_A(x)$  and  $\lambda_A^f(x) = \lambda_A(x)$  for all  $x \in R$  and  $f \in \text{Aut}(R)$ .

**Theorem 12**  $A \in \text{IFI}(R)$  is characteristic if and only if each its nonempty level set is a characteristic left  $h$ -ideal of  $R$ .

**Proof:** An IFS  $A = (\mu_A, \lambda_A)$  is an IF left  $h$ -ideal if and only if all its nonempty level subsets are left  $h$ -ideals (Theorem 9). So, we will prove only that  $A$  is characteristic if and only if all its level subsets are characteristic. If  $A = (\mu_A, \lambda_A)$  is characteristic,  $\alpha \in \text{Im}(\mu_A)$ ,  $f \in \text{Aut}(R)$ ,  $x \in U(\mu_A, \alpha)$ , then  $\mu_A^f(x) = \mu_A(f(x)) = \mu_A(x) \geq \alpha$ , which means that  $f(x) \in U(\mu_A, \alpha)$ . Thus  $f(U(\mu_A, \alpha)) \subseteq U(\mu_A, \alpha)$ . Since for each  $x \in U(\mu_A, \alpha)$  there exists  $y \in R$  such that  $f(y) = x$  we have  $\mu_A(y) = \mu_A^f(y) = \mu_A(f(y)) = \mu_A(x) \geq \alpha$ . Therefore  $y \in U(\mu_A, \alpha)$ . Thus  $x = f(y) \in f(U(\mu_A, \alpha))$ . Hence  $f(U(\mu_A, \alpha)) = U(\mu_A, \alpha)$ . Similarly,  $f(L(\lambda_A, \beta)) = L(\lambda_A, \beta)$ . This proves that  $U(\mu_A, \alpha)$  and  $L(\lambda_A, \beta)$  are characteristic.

Conversely, if all levels of  $A = (\mu_A, \lambda_A)$  are characteristic left  $h$ -ideals of  $R$ , then for  $x \in R$ ,  $f \in \text{Aut}(R)$  and  $\mu_A(x) = \alpha$ ,  $\lambda_A(x) = \beta$ , by Lemma 9, we have  $x \in U(\mu_A, \alpha)$ ,  $x \notin U(\mu_A, \gamma)$  and  $x \in L(\lambda_A, \beta)$ ,  $x \notin L(\lambda_A, \delta)$  for all  $\gamma > \alpha$ ,  $\delta < \beta$ . Thus  $f(x) \in f(U(\mu_A, \alpha)) = U(\mu_A, \alpha)$  and  $f(x) \in f(L(\lambda_A, \beta)) = L(\lambda_A, \beta)$ , i.e.,  $\mu_A(f(x)) \geq \alpha$  and  $\lambda_A(f(x)) \leq \beta$ . For  $\mu_A(f(x)) = \gamma > \alpha$ ,  $\lambda_A(f(x)) = \delta < \beta$  we have  $f(x) \in U(\mu_A, \gamma) = f(U(\mu_A, \gamma))$ ,  $f(x) \in L(\lambda_A, \delta) = f(L(\lambda_A, \delta))$ , which implies  $x \in U(\mu_A, \gamma)$ ,  $x \in L(\mu_A, \delta)$ . This is a contradiction. Thus  $\mu_A(f(x)) = \mu_A(x)$  and  $\lambda_A(f(x)) = \lambda_A(x)$ . So,  $A = (\mu_A, \lambda_A)$  is characteristic.  $\square$

**Proposition 13** Let  $f : R \rightarrow S$  be a homomorphism of hemirings. If  $A = (\mu_A, \lambda_A)$  is an IF left  $h$ -ideal of  $S$ , then  $A^f = (\mu_A^f, \lambda_A^f)$  is an IF left  $h$ -ideal of  $R$ .

**Proof:** Let  $x, y \in R$ . Then

$$\begin{aligned}\mu_A^f(x + y) &= \mu_A(f(x + y)) = \mu_A(f(x) + f(y)) \\ &\geq \min\{\mu_A(f(x)), \mu_A(f(y))\} \\ &= \min\{\mu_A^f(x), \mu_A^f(y)\},\end{aligned}$$

$$\begin{aligned}\lambda_A^f(x + y) &= \lambda_A(f(x + y)) = \lambda_A(f(x) + f(y)) \\ &\leq \max\{\lambda_A(f(x)), \lambda_A(f(y))\} \\ &= \max\{\lambda_A^f(x), \lambda_A^f(y)\},\end{aligned}$$

$$\begin{aligned}\mu_A^f(xy) &= \mu_A(f(xy)) = \mu_A(f(x)f(y)) \\ &\geq \mu_A(f(y)) = \mu_A^f(y),\end{aligned}$$

$$\begin{aligned}\lambda_A^f(xy) &= \lambda_A(f(xy)) = \lambda_A(f(x)f(y)) \\ &\leq \lambda_A(f(y)) = \lambda_A^f(y).\end{aligned}$$

If  $x + a + z = b + z$ , then  $f(x) + f(a) + f(z) = f(b) + f(z)$ , whence

$$\begin{aligned}\mu_A^f(x) &= \mu_A(f(x)) \geq \min\{\mu_A(f(a)), \mu_A(f(b))\} \\ &= \min\{\mu_A^f(a), \mu_A^f(b)\}, \\ \lambda_A^f(x) &= \lambda_A(f(x)) \leq \max\{\lambda_A(f(a)), \lambda_A(f(b))\} \\ &= \max\{\lambda_A^f(a), \lambda_A^f(b)\}.\end{aligned}$$

This proves that  $A^f = (\mu_A^f, \lambda_A^f)$  is an IF left  $h$ -ideal of  $R$ .  $\square$

**Proposition 14** Let  $f : R \rightarrow S$  be an epimorphism of hemirings. If  $A^f = (\mu_A^f, \lambda_A^f)$  is an IF left  $h$ -ideal of  $R$ , then  $A = (\mu_A, \lambda_A)$  is an IF left  $h$ -ideal of  $S$ .

**Proof:** Since  $f$  is a surjective mapping, for  $x, y \in S$  there are  $x_1, y_1 \in R$  such that  $x = f(x_1)$ ,  $y = f(y_1)$ . Thus

$$\begin{aligned}\mu_A(x + y) &= \mu_A(f(x_1) + f(y_1)) = \mu_A(f(x_1 + y_1)) \\ &= \mu_A^f(x_1 + y_1) \geq \min\{\mu_A^f(x_1), \mu_A^f(y_1)\} \\ &= \min\{\mu_A(x), \mu_A(y)\},\end{aligned}$$

proves that  $\mu_A$  satisfies the first condition on Definition 1. In a similar way we can verify others conditions.  $\square$

As a consequence of the above two propositions we obtain the following theorem.

**Theorem 15** Let  $f : R \rightarrow S$  be an epimorphism of hemirings. Then  $A^f = (\mu_A^f, \lambda_A^f)$  is an IF left  $h$ -ideal of  $R$  if and only if  $A = (\mu_A, \lambda_A)$  is an IF left  $h$ -ideal of  $S$ .

## 5 Normal IF left $h$ -ideals

**Definition 16** An IF left  $h$ -ideal  $A = (\mu_A, \lambda_A)$  of a hemiring  $R$  is said to be *normal* if  $A(0) = (1, 0)$ , i.e.,  $\mu_A(0) = 1$  and  $\lambda_A(0) = 0$ .

It is clear that any IF left  $h$ -ideal containing a normal IF left  $h$ -ideal is normal too.

**Theorem 17** Let  $A = (\mu_A, \lambda_A) \in IFI(R)$  and let  $\mu_A^+(x) = \mu_A(x) + 1 - \mu_A(0)$ ,  $\lambda_A^+(x) = \lambda_A(x) - \lambda_A(0)$ . If  $\mu_A^+(x) + \lambda_A^+(x) \leq 1$  for all  $x \in R$ , then  $A^+ = (\mu_A^+, \lambda_A^+)$  is a normal IF left  $h$ -ideal of  $R$  containing  $A$ .

**Proof:** At first observe that  $\mu_A^+(0) = 1$ ,  $\lambda_A^+(0) = 0$  and  $\mu^+(x), \lambda^+(x) \in [0, 1]$  for every  $x \in R$ . So,  $A^+ = (\mu_A^+, \lambda_A^+)$  is a normal IFS.

To prove that it is an IF left  $h$ -ideal let  $x, y \in R$ . Then

$$\begin{aligned}\mu_A^+(x+y) &= \mu_A(x+y) + 1 - \mu_A(0) \\ &\geq \min\{\mu_A(x), \mu_A(y)\} + 1 - \mu_A(0) \\ &= \min\{\mu_A(x) + 1 - \mu_A(0), \mu_A(y) + 1 - \mu_A(0)\} \\ &= \min\{\mu_A^+(x), \mu_A^+(y)\}, \\ \lambda_A^+(x+y) &= \lambda_A(x+y) - \lambda_A(0) \\ &\leq \max\{\lambda_A(x), \lambda_A(y)\} - \lambda_A(0) \\ &= \max\{\lambda_A(x) - \lambda_A(0), \lambda_A(y) - \lambda_A(0)\} \\ &= \max\{\lambda_A^+(x), \lambda_A^+(y)\},\end{aligned}$$

and

$$\begin{aligned}\mu_A^+(xy) &= \mu_A(xy) + 1 - \mu_A(0) \\ &\geq \mu_A(y) + 1 - \mu_A(0) = \mu_A^+(y), \\ \lambda_A^+(xy) &= \lambda_A(xy) - \lambda_A(0) \\ &\leq \lambda_A(y) - \lambda_A(0) = \lambda_A^+(y).\end{aligned}$$

This shows that  $A^+$  is an IF left ideal of  $R$ . Moreover, if  $x + a + z = b + z$ , then

$$\begin{aligned}\mu_A^+(x) &= \mu_A(x) + 1 - \mu_A(0) \\ &\geq \min\{\mu_A(a), \mu_A(b)\} + 1 - \mu_A(0) \\ &= \min\{\mu_A(a) + 1 - \mu_A(0), \mu_A(b) + 1 - \mu_A(0)\} \\ &= \min\{\mu_A^+(a), \mu_A^+(b)\}.\end{aligned}$$

Similarly

$$\begin{aligned}\lambda_A^+(x) &= \lambda_A(x) - \lambda_A(0) \\ &\leq \max\{\lambda_A(a), \lambda_A(b)\} - \lambda_A(0) \\ &= \max\{\lambda_A(a) - \lambda_A(0), \lambda_A(b) - \lambda_A(0)\} \\ &= \max\{\lambda_A^+(a), \lambda_A^+(b)\}.\end{aligned}$$

So,  $A^+ = (\mu_A^+, \lambda_A^+)$  is a normal IF left  $h$ -ideal of  $R$ . Clearly  $A \subseteq A^+$ .  $\square$

**Remark 18** In the above theorem the assumption  $\mu_A^+(x) + \lambda_A^+(x) \leq 1$  is essential. Indeed, for an IF left  $h$ -ideal  $A$  defined in Example 2 for every  $x \in R$  we have  $\mu_A^+(x) = \mu_A(x) + 0.6 \in [0, 1]$  and  $\lambda_A^+(x) = \lambda_A(x) - 0.2 \in [0, 1]$ , but  $\mu_A^+(x) + \lambda_A^+(x) > 1$  for all  $x > 0$ . So,  $A^+ = (\mu_A^+, \lambda_A^+)$  is not an IFS.

**Corollary 19**  $(A^+)^+ = A^+$  for any  $A \in IFI(R)$ . If  $A$  is normal, then  $A^+ = A$ .

Denote by  $NIFI(R)$  the set of all normal IF left  $h$ -ideals of  $R$ . Note that  $NIFI(R)$  is a poset under the set inclusion.

**Theorem 20** A non-constant maximal element of  $(NIFI(R), \subseteq)$  takes only the values  $(0, 1)$  and  $(1, 0)$ .

**Proof:** Let  $A = (\mu_A, \lambda_A) \in NIFI(R)$  be a non-constant maximal element of  $(NIFI(R), \subseteq)$ . Then  $\mu_A(0) = 1$  and  $\lambda_A(0) = 0$ . Let  $x \in R$  be such that  $\mu(x) \neq 1$ . We claim that  $\mu_A(x) = 0$ . If not, then there exists  $c \in R$  such that  $0 < \mu_A(c) < 1$ . Let  $A_c = (\nu_A, \rho_A)$  be an IFS in  $R$  defined by

$$\begin{aligned}\nu_A(x) &= \frac{1}{2}\{\mu_A(x) + \mu_A(c)\}, \\ \rho_A(x) &= \frac{1}{2}\{\lambda_A(x) + \lambda_A(c)\}\end{aligned}$$

for all  $x \in R$ . Then clearly an IFS  $A_c$  is well-defined and

$$\begin{aligned}\nu_A(0) &= \frac{1}{2}\{\mu_A(0) + \mu_A(c)\} \\ &\geq \frac{1}{2}\{\mu_A(x) + \mu_A(c)\} = \nu_A(x), \\ \rho_A(0) &= \frac{1}{2}\{\lambda_A(0) + \lambda_A(c)\} \\ &\leq \frac{1}{2}\{\lambda_A(x) + \lambda_A(c)\} = \rho_A(x)\end{aligned}$$

for all  $x \in R$ . For all  $x, y \in R$  we have also

$$\begin{aligned}\nu_A(x+y) &= \frac{1}{2}\{\mu_A(x+y) + \mu_A(c)\} \\ &\geq \frac{1}{2}\{\min\{\mu_A(x), \mu_A(y)\} + \mu_A(c)\} \\ &= \min\{\frac{1}{2}\{\mu_A(x) + \mu_A(c)\}, \frac{1}{2}\{\mu_A(y) + \mu_A(c)\}\} \\ &= \min\{\nu_A(x), \nu_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\nu_A(xy) &= \frac{1}{2}\{\mu_A(xy) + \mu_A(c)\} \\ &\geq \frac{1}{2}\{\mu_A(y) + \mu_A(c)\} = \nu_A(y).\end{aligned}$$

Analogously  $\rho_A(x+y) \leq \max\{\rho_A(x), \rho_A(y)\}$  and  $\rho_A(xy) \leq \rho_A(y)$ .

Moreover, if  $x + a + z = b + z$ , then

$$\begin{aligned}\nu_A(x) &= \frac{1}{2}\{\mu_A(x) + \mu_A(c)\} \\ &\geq \frac{1}{2}\{\min\{\mu_A(a), \mu_A(b)\} + \mu_A(c)\} \\ &= \min\{\frac{1}{2}\{\mu_A(a) + \mu_A(c)\}, \frac{1}{2}\{\mu_A(b) + \mu_A(c)\}\} \\ &= \min\{\nu_A(a), \nu_A(b)\}\end{aligned}$$

and, by analogy,  $\rho_A(x) \leq \max\{\rho_A(a), \rho_A(b)\}$ . This proves that  $A_c \in IFI(R)$ .

According to Theorem 17,  $A_c^+ = (\nu_A^+, \rho_A^+)$ , where  $\nu_A^+(x) = \nu_A(x) + 1 - \nu_A(0) = \frac{1}{2}\{1 + \mu_A(x)\}$  and  $\rho_A^+(x) = \rho_A(x) - \rho_A(0) = \frac{1}{2}\lambda_A(x)$ , belongs to  $NIFI(R)$ . Clearly  $A \subseteq A_c^+$ .

Since  $\nu_A^+(x) = \frac{1}{2}(1 + \mu_A(x)) > \mu_A(x)$ ,  $A$  is a proper subset of  $A_c^+$ . Obviously  $\nu_A^+(a) < 1 = \nu_A^+(0)$ . Hence  $A_c^+$  is non-constant, and  $A$  is not a maximal element of  $NIFI(R)$ . This is a contradiction. Therefore  $\mu_A$  takes only two values: 0 and 1.

Analogously we can prove that  $\lambda_A$  also takes the values 0 and 1. This means that for  $A$  the possible values are (0,0), (0,1) and (1,0). If  $A$  takes these three values, then

$$\begin{aligned} R^{(0,0)} &= \{x \in R \mid \mu_A(x) \geq 0, \lambda_A(x) \leq 0\} \\ &= \{x \in R \mid \lambda_A(x) = 0\}, \\ R^{(1,0)} &= \{x \in R \mid \mu_A(x) \geq 1, \lambda_A(x) \leq 0\} \\ &= \{x \in R \mid \mu_A(x) = 1, \lambda_A(x) = 0\}, \\ R^{(0,1)} &= \{x \in R \mid \mu_A(x) \geq 0, \lambda_A(x) \leq 1\} = R \end{aligned}$$

are nonempty left  $h$ -ideals (Theorem 8) such that  $R^{(1,0)} \subset R^{(0,0)} \subset R^{(0,1)} = R$ . Then, according to Proposition 6, an IFS  $B = (\mu_B, \lambda_B)$  defined by

$$\begin{aligned} \mu_B(x) &= \begin{cases} 1 & \text{if } x \in R^{(0,0)} \\ 0 & \text{if } x \notin R^{(0,0)} \end{cases} \\ \lambda_B(x) &= \begin{cases} 0 & \text{if } x \in R^{(0,0)} \\ 1 & \text{if } x \notin R^{(0,0)} \end{cases} \end{aligned}$$

is an intuitionistic fuzzy left  $h$ -ideals of  $R$ . It is normal. Moreover,  $\lambda_A(x) \neq 0$  for  $x \in R \setminus R^{(0,0)}$ . Thus  $\lambda_A(x) = 1$ , consequently  $\mu_A(x) = 0$ . This implies  $A(x) = B(x)$  for  $x \in R \setminus R^{(0,0)}$ . For  $x \in R^{(0,0)}$  we have  $\lambda_A(x) = 0 = \lambda_B(x)$  and  $\mu_A(x) \leq 1 = \mu_B(x)$ . Hence  $A \subset B$ . Since  $\mu_A(x) = 0 < \mu_B(x)$  for  $x \in R^{(0,0)} \setminus R^{(1,0)}$ , an IF left  $h$ -ideal  $A$  is a proper subset of  $B$ . This is a contradiction. So, a non-constant maximal element of  $(NIFI(R), \subseteq)$  takes only two values: (0,1) and (1,0).  $\square$

**Definition 21** A non-constant  $A \in IFI(R)$  is called *maximal* if  $A^+$  is a maximal element of the poset  $(NIFI(R), \subseteq)$ .

**Theorem 22** A maximal  $A \in IFI(R)$  is normal and takes only two values: (0,1) and (1,0).

**Proof:** Let  $A \in IFI(R)$  be maximal. Then  $A^+$  is a non-constant maximal element of  $(NIFI(R), \subseteq)$  and, by Theorem 20, the possible values of  $A^+$  are

(0,1) and (1,0), i.e.,  $\mu_A^+$  takes only two values 0 and 1. Clearly  $\mu_A^+(x) = 1$  if and only if  $\mu_A(x) = \mu_A(0)$ , and  $\mu_A^+(x) = 0$  if and only if  $\mu_A(x) = \mu_A(0) - 1$ . But  $A \subseteq A^+$  (Theorem 17), so,  $\mu_A(x) \leq \mu_A^+(x)$  for all  $x \in R$ . Thus  $\mu_A^+(x) = 0$  implies  $\mu_A(x) = 0$ , whence  $\mu_A(0) = 1$ . This proves that  $A$  is normal.  $\square$

**Theorem 23** A (1,0)-level subset of a maximal IF left  $h$ -ideal of  $R$  is a maximal left  $h$ -ideal of  $R$ .

**Proof:** Let  $S$  be a (1,0)-level subset of a maximal  $A \in IFI(R)$ , i.e.,

$$S = R^{(1,0)} = \{x \in R \mid \mu_A(x) = 1\}.$$

It is not difficult to verify that  $S$  is a left  $h$ -ideal of  $R$ .  $S \neq R$  because  $\mu_A$  takes two values.

Let  $M$  be a left  $h$ -ideal of  $R$  containing  $S$ . Then  $\mu_S \subseteq \mu_M$ . Since  $\mu_A = \mu_S$  and  $\mu_A$  takes only two values,  $\mu_M$  also takes these two values. But, by the assumption,  $A \in IFI(R)$  is maximal, so  $\mu_S = \mu_A = \mu_M$  or  $\mu_M(x) = 1$  for all  $x \in R$ . In the last case  $S = R$ , which is impossible. So,  $\mu_A = \mu_S = \mu_M$ , which implies  $S = M$ . This means that  $S$  is a maximal left  $h$ -ideal of  $R$ .  $\square$

**Definition 24** A normal  $A \in IFI(R)$  is called *completely normal* if there exists  $x_0 \in R$  such that  $A(x_0) = (0,1)$ .

Denote by  $\mathcal{C}(R)$  the set of all completely normal  $A \in IFI(R)$ . Clearly  $\mathcal{C}(R) \subseteq NIFI(R)$ .

**Theorem 25** A non-constant maximal element of  $(NIFI(R), \subseteq)$  is also a maximal element of  $(\mathcal{C}(R), \subseteq)$ .

**Proof:** Let  $A$  be a non-constant maximal element of  $(NIFI(R), \subseteq)$ . By Theorem 20,  $A$  takes only the values (0,1) and (1,0), so  $A(0) = (1,0)$  and  $A(x_0) = (0,1)$  for some  $x_0 \in R$ . Hence  $A \in \mathcal{C}(R)$ . Assume that there exists  $B \in \mathcal{C}(R)$  such that  $A \subseteq B$ . It follows that  $A \subseteq B$  in  $NIFI(R)$ . Since  $A$  is maximal in  $(NIFI(R), \subseteq)$  and since  $B$  is non-constant, therefore  $A = B$ . Thus  $A$  is maximal element of  $(\mathcal{C}(R), \subseteq)$ , ending the proof.  $\square$

**Theorem 26** Every maximal  $A \in IFI(R)$  is completely normal.

**Proof:** Let  $A \in IFI(R)$  be maximal. Then by Theorem 22, it is normal and  $A = A^+$  takes only two values (0,1) and (1,1). Since  $A$  is non-constant, it follows that  $A(0) = (1,0)$  and  $A(x_0) = (0,1)$  for some  $x_0 \in R$ . Hence  $A$  is completely normal, ending the proof.  $\square$

Below we present the method of construction a new normal intuitionistic fuzzy left  $h$ -ideal from old.

**Theorem 27** Let  $f : [0, 1] \rightarrow [0, 1]$  be an increasing function and let  $A = (\mu_A, \lambda_A)$  be an IFS on a hemiring  $R$ . Then  $A_f = (\mu_{A_f}, \lambda_{A_f})$ , where  $\mu_{A_f}(x) = f(\mu_A(x))$  and  $\lambda_{A_f}(x) = f(\lambda_A(x))$ , is an IF left  $h$ -ideal if and only if  $A = (\mu_A, \lambda_A)$  is an IF left  $h$ -ideal. Moreover, if  $f(\mu_A(0)) = 1$  and  $f(\lambda_A(0)) = 0$ , then  $A_f$  is normal.

**Proof:** We will verify only the condition (1). Let  $A_f = (\mu_{A_f}, \lambda_{A_f}) \in IFI(R)$ . Then

$$\begin{aligned} f(\mu_A(x+y)) &= \mu_{A_f}(x+y) \\ &\geq \min\{\mu_{A_f}(x), \mu_{A_f}(y)\} \\ &= \min\{f(\mu_A(x)), f(\mu_A(y))\} \\ &= f(\min\{\mu_A(x), \mu_A(y)\}), \end{aligned}$$

i.e.,

$$f(\mu_A(x+y)) \geq f(\min\{\mu_A(x), \mu_A(y)\})$$

for all  $x, y \in R$ . Since  $f$  is increasing, must be

$$\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}.$$

Conversely, if  $A = (\mu_A, \lambda_A) \in IFI(R)$ , then for all  $x, y \in R$  we have

$$\begin{aligned} \mu_{A_f}(x+y) &= f(\mu_A(x+y)) \\ &\geq f(\min\{\mu_A(x), \mu_A(y)\}) \\ &= \min\{f(\mu_A(x)), f(\mu_A(y))\} \\ &= \min\{\mu_{A_f}(x), \mu_{A_f}(y)\}, \end{aligned}$$

i.e.,  $\mu_{A_f}(x+y) \geq \min\{\mu_{A_f}(x), \mu_{A_f}(y)\}$ .

This proves that  $A_f = (\mu_{A_f}, \lambda_{A_f})$  satisfies (1) if and only if it is satisfying by  $A = (\mu_A, \lambda_A)$ .

In the same manner we can prove the analogous connections for the axioms (2) – (6).  $\square$

## 6 Conclusion

In the present paper we present the basic results on IF left  $h$ -ideals of hemirings. In our opinion the future study of different types of IF ideals in hemirings and near rings can be connected with (1) investigating semiprime and prime IF  $h$ -ideals; (2) finding intuitionistic and/or interval valued fuzzy sets and triangular norms. The obtained results can be used to solve some social networks problems, automata theory and formal languages.

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